EXPRESSION OF UNITARY COMPONENTS OF THE HIGHEST ORDER INTERACTIONS IN 3^5, 3^6, 4^4 AND 5^3 DESIGNS IN TERMS OF SETS FOR THESE INTERACTIONS

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1. Introduction

In a previous paper (Kishen, 1942), a general method was developed for expressing any single degree of freedom for treatments in the case of the general symmetrical factorial design s^m, s being a prime positive integer or a power of a prime and m any positive integer, in terms of its sets for main effects and interactions, and was utilized for obtaining expressions for the unitary components of the third order interaction in a 3^4 design and of the second order interaction in a 4^3 design. When, however, the single degree of freedom belongs to a $(k-1)$-th order interaction (k varying from 1 to m), a simplified and short-cut method of deriving these expressions has been described in the present paper and has been employed for deriving expressions for the unitary components of the highest order interactions in the 3^5, 3^6, 4^4 and 5^3 designs. Throughout this paper, when dealing with the finite elements of the m-dimensional finite projective geometry $PG(m, s)$, we shall as usual write their co-ordinates, equations, etc., as if they belonged to the m-dimensional finite Euclidean geometry $EG(m, s)$ immersed in the projective geometry (Bose and Kishen, 1940).

2. Method of Obtaining Expressions for any Single Degree of Freedom Belonging to the $(k-1)$-th Order Interaction in an s^m Design

In an s^m design, let any treatment combination (or the quantitative measure of the result of application of the treatment combination) be represented by the symbol $a_1^{i_1} a_2^{i_2} \ldots a_m^{i_m}$, where $a_r^{i_r}$ denotes the i_r-th level of the r-th factor (i_r varying from 0 to $s-1$, and r varying from 1 to m). Then any single degree of freedom belonging to treatments may be written as

$$L = \Sigma\, l_{i_1 i_2 \ldots i_m} a_1^{i_1} a_2^{i_2} \ldots a_m^{i_m} \quad (i_1, i_2, \ldots, i_m \text{ varying from } 0 \text{ to } s-1),$$

where $l_{i_1 i_2 \ldots i_m}$ is a constant coefficient such that $\Sigma l_{i_1 i_2 \ldots i_m} = 0$.

land records and agricultural departments, the total number of villages selected for sampling are divided into two random groups, one group being assigned to the staff of one department and the other to the staff of the other department. As in both cases the staff works within the area under their normal jurisdiction, no special travelling is involved and the cost of the survey is not affected. Without, therefore, affecting the efficiency of the survey, this sort of replication can provide information on the relative efficiency with which the two agencies carry out the field work.

SUMMARY

The method of interpenetrating samples is a design for the sample surveys in which the sample units are arranged in sets of two or more interpenetrating samples and the information for each set is collected in an independent manner. Mahalanobis has used this design in the area surveys he carried out in Bengal and Bihar as a means of controlling the reliability of field work. The statistical efficiency of the design in relation to the precision of the estimate and the cost involved, has been examined in the present note. It has been shown that the method leads to an appreciable loss of information per unit of cost. This loss for the jute area survey in Bengal is computed at 21 per cent. In more extreme cases nearly half of the information may be lost. In the case where the sample units are independently located at random and are then grouped into two sub-samples, the loss of information per unit of cost would still be 8 to 17 per cent.

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It has been demonstrated in the previous paper that the expression of L in terms of sets for main effects and interactions for the unreplicated case is given by

$$L = \frac{1}{s^{m-1}} \Sigma l_{i_1 i_2 \ldots i_m} \left[\text{Sum of the } \frac{s^m - 1}{s - 1} \text{ sets for main effects and interactions containing } a_1^{i_1} a_2^{i_2} \ldots a_m^{i_m} \right] \quad (1)$$

The practical procedure of obtaining the expressions for any single degree of freedom has been outlined in the previous paper. When, however, the single degree of freedom belongs to a $(k - 1)$-th order interaction $A_{u_1} A_{u_2} \ldots A_{u_k}$ ($u_1, u_2, \ldots, u_k$ being some k of the integers $1, 2, \ldots, m$), the expression for it will involve only the sets for the interaction $A_{u_1} A_{u_2} \ldots A_{u_k}$ and is easily derived by the simplified procedure described in the subsequent paragraphs.

It is well known that the expression for a single degree of freedom belonging to the highest order interaction $A_{u_1} A_{u_2} \ldots A_{u_k}$ in the factorial design s^k formed by all combinations of the u_1-th, u_2-th, $\ldots$, and u_k-th factors out of m factors of the factorial design s^m, $(k \leqslant m)$, is of exactly the same form as the expression for this single degree of freedom when considered as belonging to the $(k - 1)$-th order interaction $A_{u_1} A_{u_2} \ldots A_{u_k}$ in an s^m design. When, therefore, the expression for a single degree of freedom belonging to a $(k - 1)$-th order interaction $A_{u_1} A_{u_3} \ldots A_{u_k}$ of an s^m design is required, it would do to work out the expression for the single degree of freedom in the s^k design formed by all combinations of the u_1-th, u_2-th, $\ldots$, and u_k-th, factors, considering this single degree of freedom as belonging to the highest order interaction of this design.

In the two-way table for treatment combinations and sets for main effects and interactions for the s^k design, only the sets corresponding to the $(s - 1)^{k-1}$ parallel pencils of $(k - 1)$-flats in $PG(k, s)$ given by

$$x_{u_1} + \alpha_{i_2} x_{u_2} + \ldots + \alpha_{i_k} x_{u_k} = \alpha_t \ (u_1, u_2, \ldots, u_k \text{ fixed}; \ i_2, \ldots, i_k = 1, 2, \ldots, s - 1; \ t = 0, 1, \ldots, s - 1) \quad (2)$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{s-1}$ are the s elements of $GF(s)$, for the highest order interaction of the s^k design need be entered instead of sets for all the main effects and interactions of this design. The treatment combinations are written down in the usual systematic manner in the column on the extreme left, as shown in Table I in the case of the 4^4 design, and on the top of the subsequent $(s - 1)^{k-1}$ columns of the table are entered systematically the left-hand sides of the equations of the $(s-1)^{k-1}$ pencils in (2) (*vide* Table I). Under each of these columns, each row is filled in by the number t (varying from 0 to $s-1$), where t denotes the element α_t of $GF(s)$ given by

$$\alpha_t = x'_{u_1} + \alpha_{i_2} x'_{u_2} + \ldots + \alpha_{i_k} x'_{u_k} \quad (i_2, \ldots, i_k \text{ fixed}) \tag{3}$$

$(x'_{u_1}, x'_{u_2}, \ldots, x'_{u_k})$ being the finite point in $PG(k, s)$ corresponding to the treatment combination on the extreme left of the horizontal row.

The two-way table may now be dealt with in s^{k-2} portions, each containing s^2 treatment combinations corresponding to all combinations of the first two, *viz.*, u_1-th and u_2-th factors. Consider the sets corresponding to the first $s-1$ pencils in (2), the left-hand sides of the equations of which are given by

$$\left.\begin{array}{l}
x_{u_1} + \alpha_1 x_{u_2} + \alpha_1 x_{u_3} + \alpha_1 x_{u_4} + \ldots + \alpha_1 x_{u_r} + \ldots + \alpha_1 x_{u_k} \\
x_{u_1} + \alpha_2 x_{u_2} + \alpha_1 x_{u_3} + \alpha_1 x_{u_4} + \ldots + \alpha_1 x_{u_r} + \ldots + \alpha_1 x_{u_k} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_{u_1} + \alpha_p x_{u_2} + \alpha_1 x_{u_3} + \alpha_1 x_{u_4} + \ldots + \alpha_1 x_{u_r} + \ldots + \alpha_1 x_{u_k} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_{u_1} + \alpha_{s-1} x_{u_2} + \alpha_1 x_{u_3} + \alpha_1 x_{u_4} + \ldots + \alpha_1 x_{u_r} + \ldots + \alpha_1 x_{u_k}
\end{array}\right\} \tag{4}$$

The remaining $(s-1)^{k-1} - (s-1)$ pencils are divisible into $(s-1)^{k-2} - 1$ groups, each of $s-1$ pencils, each of these groups being similar to that in (4), the left-hapd sides of the equations of the $s-1$ pencils in the general group being given by

$$\left.\begin{array}{l}
x_{u_1} + \alpha_1 x_{u_2} + \alpha_{i_3} x_{u_3} + \alpha_{i_4} x_{u_4} + \ldots + \alpha_{i_r} x_{u_r} + \ldots + \alpha_{i_k} x_{u_k} \\
x_{u_1} + \alpha_2 x_{u_2} + \alpha_{i_3} x_{u_3} + \alpha_{i_4} x_{u_4} + \ldots + \alpha_{i_r} x_{u_r} + \ldots + \alpha_{i_k} x_{u_k} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_{u_2} + \alpha_p x_{u_2} + \alpha_{i_3} x_{u_3} + \alpha_{i_4} x_{u_4} + \ldots + \alpha_{i_r} x_{u_r} + \ldots + \alpha_{i_k} x_{u_k} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_{u_1} + \alpha_{s-1} x_{u_2} + \alpha_{i_3} x_{u_3} + \alpha_{i_4} x_{u_4} + \ldots + \alpha_{i_r} x_{u_r} + \ldots + \alpha_{i_k} x_{u_k}
\end{array}\right\} \tag{5}$$

Then the difference between any two corresponding expressions in (4) and (5) comes out to be

$$(\alpha_{i_3} - \alpha_1)\, x_{u_3} + (\alpha_{i_4} - \alpha_1)\, x_{u_4} + \ldots + (\alpha_{i_r} - \alpha_1)\, x_{u_r} + \ldots + (\alpha_{i_k} - \alpha_1)\, x_{u_k} \tag{6}$$

which remains constant in each of the above s^{k-2} portions of s^2 treatment combinations, since $x_{u_3}, x_{u_4}, \ldots, x_{u_k}$ are the same for each of the s^2 treatment combinations in each of the s^{k-2} portions. Consequently, the constant coefficients corresponding to the given single degree of freedom of the highest order interaction in the s^k design need only be totalled up for the sets corresponding to the first group of $s-1$ pencils, the totals of the coefficients for the sets corresponding to the remaining $(s-1)^{k-2}-1$ groups of $s-1$ pencils being readily written down from the totals for the first group in view of the constancy of difference in the corresponding expressions in (4) and (5) established above. Thus, over the entire two-way table, the totalling up of the coefficients has to be done for only the sets corresponding to the first group of $s-1$ pencils, in consequence of which the computational labour is reduced to about $1/(s-1)^{k-2}$ of what would be required in the general procedure given in the previous paper. An actual example will further elucidate the procedure.

(2.1) Illustration of the above Procedure: Expression for A′B′C′D′ in a 4^4 Design

Let us consider a 4^4 design, *viz.*,

$$(a_0, a_1, a_2, a_3) \times (b_0, b_1, b_2, b_3) \times (c_0, c_1, c_2, c_3) \times (d_0, d_1, d_2, d_3),$$

and suppose it is required to obtain the expression for the single degree of freedom given by

$$A'B'C'D' = (a_3 + a_2 - a_1 - a_0)(b_3 + b_2 - b_1 - b_0)(c_3 + c_2 - c_1 - c_0)(d_3 + d_2 - d_1 - d_0) \tag{7}$$

In this case there are 16 portions, each containing 16 treatment combinations corresponding to all combinations of the first two factors, of the two-way table of treatment combinations and 27 pencils of 3-flats in $PG(4, 4)$ given by

$$x_1 + \alpha_{i_2}x_2 + \alpha_{i_3}x_3 + \alpha_{i_4}x_4 = \alpha_t \ (i_2, i_3, i_4 = 1, 2, 3;\ t = 0, 1, 2, 3) \tag{8}$$

where $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2 = x$ and $\alpha_3 = 1 + x$ are the 4 elements of $GF(2^2)$. The sets of third order interaction corresponding to these pencils, written down in order in the usual systematic manner, may be symbolized by $R_1, R_2, R_3, \ldots, R_{27}$ respectively.

In Table I is given the last portion of this two-way table comprised of 16 treatment combinations. The constant coefficients corresponding

TABLE I

Last portion of two-way table for treatment combinations and 27 sets of third order interaction in a 4^4 design

Treatment combinations \ Sets				$x_1+x_2+x_3+x_4$	$x_1+2x_2+x_3+x_4$	$x_1+3x_2+x_3+x_4$	$x_1+x_2+2x_3+x_4$	$x_1+2x_2+2x_3+x_4$	$x_1+3x_2+2x_3+x_4$	$x_1+x_2+3x_3+x_4$	$x_1+2x_2+3x_3+x_4$	$x_1+3x_2+3x_3+x_4$	$x_1+x_2+x_3+2x_4$	$x_1+2x_2+x_3+2x_4$	$x_1+3x_2+x_3+2x_4$	$x_1+x_2+2x_3+2x_4$	$x_1+2x_2+2x_3+2x_4$	$x_1+3x_2+2x_3+2x_4$
a	*b*	*c*	*d*	(R_1)	(R_2)	(R_3)	(R_4)	(R_5)	(R_6)	(R_7)	(R_8)	(R_9)	(R_{10})	(R_{11})	(R_{12})	(R_{13})	(R_{14})	(R_{15})
0	0	3	3	0	0	0	2	2	2	1	1	1	2	2	2	0	0	0
1	0	3	3	1	1	1	3	3	3	0	0	0	3	3	3	1	1	1
2	0	3	3	2	2	2	0	0	0	3	3	3	0	0	0	2	2	2
3	0	3	3	3	3	3	1	1	1	2	2	2	1	1	1	3	3	3
0	1	3	3	1	2	3	3	0	1	0	3	2	3	0	1	1	2	3
1	1	3	3	0	3	2	2	1	0	1	2	3	2	1	0	0	3	2
2	1	3	3	3	0	1	1	2	3	2	1	0	1	2	3	3	0	1
3	1	3	3	2	1	0	0	3	2	3	0	1	0	3	2	2	1	0
0	2	3	3	2	3	1	0	1	3	3	2	0	0	1	3	2	3	1
1	2	3	3	3	2	0	1	0	2	2	3	1	1	0	2	3	2	0
2	2	3	3	0	1	3	2	3	1	1	0	2	2	3	1	0	1	3
3	2	3	3	1	0	2	3	2	0	0	1	3	3	2	0	1	0	2
0	3	3	3	3	1	2	1	3	0	2	0	3	1	3	0	3	1	2
1	3	3	3	2	0	3	0	2	1	3	1	2	0	2	1	2	0	3
2	3	3	3	1	3	0	3	1	2	0	2	1	3	1	2	1	3	0
3	3	3	3	0	2	1	2	0	3	1	3	0	2	0	3	0	2	1

TABLE I—(*Contd.*)

Last portion of two-way table for treatment combinations and 27 sets of third order interaction in a 4^4 design

Treatment combinations \ Sets				$x_1+x_2+3x_3+2x_4$	$x_1+2x_2+3x_3+2x_4$	$x_1+3x_2+3x_3+2x_4$	$x_1+x_2+x_3+3x_4$	$x_1+2x_2+x_3+3x_4$	$x_1+3x_2+x_3+3x_4$	$x_1+x_2+2x_3+3x_4$	$x_1+2x_2+2x_3+3x_4$	$x_1+3x_2+2x_3+3x_4$	$x_1+x_2+3x_3+3x_4$	$x_1+2x_2+3x_3+3x_4$	$x_1+3x_2+3x_3+3x_4$	Constant coefficients for $A'B'C'D'$
a	b	c	d	(R_{16})	(R_{17})	(R_{18})	(R_{19})	(R_{20})	(R_{21})	(R_{22})	(R_{23})	(R_{24})	(R_{25})	(R_{26})	(R_{27})	
0	0	3	3	3	3	3	1	1	1	3	3	3	0	0	0	+1
1	0	3	3	2	2	2	0	0	0	2	2	2	1	1	1	+1
2	0	3	3	1	1	1	3	3	3	1	1	1	2	2	2	−1
3	0	3	3	0	0	0	2	2	2	0	0	0	3	3	3	−1
0	1	3	3	2	1	0	0	3	2	2	1	0	1	2	3	+1
1	1	3	3	3	0	1	1	2	3	3	0	1	0	3	2	+1
2	1	3	3	0	3	2	2	1	0	0	3	2	3	0	1	−1
3	1	3	3	1	2	3	3	0	1	1	2	3	2	1	0	−1
0	2	3	3	1	0	2	3	2	0	1	0	2	2	3	1	−1
1	2	3	3	0	1	3	2	3	1	0	1	3	3	2	0	−1
2	2	3	3	3	2	0	1	0	2	3	2	0	0	1	3	+1
3	2	3	3	2	3	1	0	1	3	2	3	1	1	0	2	+1
0	3	3	3	0	2	1	2	0	3	0	2	1	3	1	2	−1
1	3	3	3	1	3	0	3	1	2	1	3	0	2	0	3	−1
2	3	3	3	2	0	3	0	2	1	2	0	3	1	3	0	+1
3	3	3	3	3	1	2	1	3	0	3	1	2	0	2	1	+1

to the single degree of freedom (7) are shown against each treatment combination in the last column in Table I. Evidently, there are here 9 groups of 3 pencils each, and we have to total up the coefficients for only the first group of 3 pencils, the left-hand sides of which are given by

$$x_1 + x_2 + x_3 + x_4,\ x_1 + 2x_2 + x_3 + x_4 \text{ and } x_1 + 3x_2 + x_3 + x_4, \tag{9}$$

0, 1, 2, 3 denoting the 4 elements α_0, α_1, α_2 and α_3 of $GF(2^2)$. These pencils, as stated above, correspond to the sets R_1, R_2 and R_3 respectively of the third order interaction in this case. We shall denote the 4 sub-sets of the first set corresponding to the four 3-flats of the first pencil $x_1 + x_2 + x_3 + x_4 = 0, 1, 2, 3$ by R_{11}, R_{12}, R_{13}, R_{14} respectively, and similarly for the remaining 26 sets.

Now, accumulating the constant coefficients, corresponding to the above single degree of freedom, for the first group of three sets corresponding to the first group of three pencils (9), we obtain totals shown below in Table II.

TABLE II

Totals of coefficients for the first group of three sets

Sub-set \ Set	$x_1+x_2+x_3+x_4$ (R_1)	$x_1+2x_2+x_3+x_4$ (R_2)	$x_1+3x_2+x_3+x_4$ (R_3)
First (0)	4	0	0
Second (1)	4	0	0
Third (2)	−4	0	0
Fourth (3)	−4	0	0

It would be seen from Table I that the totals of the coefficients for the first sub-sets of the sets of the second group are respectively exactly the same as the third sub-sets of sets of the first group, *viz.*, −4, 0, 0. Similarly, the totals of the coefficients for the first sub-sets of sets belonging to the remaining groups may be directly written down, as also those for the second, third and fourth sub-sets of sets belonging to the eight groups other than the first. The required totals are shown in Table III.

Adding up the totals of coefficients derived by this procedure for the remaining 15 portions of the two-way table, we finally obtain

$$A'B'C'D' = R_{11} + R_{12} - R_{13} - R_{14} \tag{10}$$

TABLE III

Totals of coefficients for the 27 sets, considered in 9 groups of three

Set / Sub-set	$x_1+x_2+x_3+x_4$	$x_1+2x_2+x_3+x_4$	$x_1+3x_2+x_3+x_4$	$x_1+x_2+2x_3+x_4$	$x_1+2x_2+2x_3+x_4$	$x_1+3x_2+2x_3+x_4$	$x_1+x_2+3x_3+x_4$	$x_1+2x_2+3x_3+x_4$	$x_1+3x_2+3x_3+x_4$	$x_1+x_2+x_3+2x_4$	$x_1+2x_2+x_3+2x_4$	$x_1+3x_2+x_3+2x_4$	$x_1+x_2+2x_3+2x_4$	$x_1+2x_2+2x_3+2x_4$	$x_1+3x_2+2x_3+2x_4$
	(R_1)	(R_2)	(R_3)	(R_4)	(R_5)	(R_6)	(R_7)	(R_8)	(R_9)	(R_{10})	(R_{11})	(R_{12})	(R_{13})	(R_{14})	(R_{15})
First (0) ..	4	0	0	−4	0	0	4	0	0	−4	0	0	4	0	0
Second (1) ..	4	0	0	−4	0	0	4	0	0	−4	0	0	4	0	0
Third (2) ..	−4	0	0	4	0	0	−4	0	0	4	0	0	−4	0	0
Fourth (3) ..	−4	0	0	4	0	0	−4	0	0	4	0	0	−4	0	0

TABLE III—(*Contd.*)

Totals of coefficients for the 27 sets, considered in 9 groups of three

Sub-set \ Set	$x_1+x_2+3x_3+2x_4$	$x_1+2x_2+3x_3+2x_4$	$x_1+3x_2+3x_3+2x_4$	$x_1+x_2+x_3+3x_4$	$x_1+2x_2+x_3+3x_4$	$x_1+3x_2+x_3+3x_4$	$x_1+x_2+2x_3+3x_4$	$x_1+2x_2+2x_3+3x_4$	$x_1+3x_2+2x_3+3x_4$	$x_1+x_2+3x_3+3x_4$	$x_1+2x_2+3x_3+3x_4$	$x_1+3x_2+3x_3+3x_4$
	(R_{16})	(R_{17})	(R_{18})	(R_{19})	(R_{20})	(R_{21})	(R_{22})	(R_{23})	(R_{24})	(R_{25})	(R_{26})	(R_{27})
First (0) ..	−4	0	0	4	0	0	−4	0	0	4	0	0
Second (1) ..	−4	0	0	4	0	0	−4	0	0	4	0	0
Third (2) ..	4	0	0	−4	0	0	4	0	0	−4	0	0
Fourth (3) ..	4	0	0	−4	0	0	4	0	0	−4	0	0

3. Maximum Number of Types of Unitary Components of a $(k-1)$-th Order Interaction in an s^m Design

Two or more unitary components of a $(k-1)$-th order interaction in an s^m design will be defined as belonging to the same type if their expressions in terms of sets for the $(k-1)$-th order interaction are exactly similar. For example, in Table 3 of the previous paper showing expressions for the components of the third order interaction in a 3^4 design, the four components $A_2B_1C_1D_1$, $A_1B_2C_1D_1$, $A_1B_1C_2D_1$ and $A_1B_1C_1D_2$ are of the same type, which may conveniently be represented by the set of suffixes 2111, the number of components belonging to this type being evidently the number of permutations of these suffixes. Similarly, the six components $A_2B_2C_1D_1$, $A_2B_1C_2D_1$, $A_1B_2C_2D_1$, $A_2B_1C_1D_2$, $A_1B_2C_1D_2$ and $A_1B_1C_2D_2$ also belong to the same type 2211.

Since the suffixes correspond to the splitting up of a main effect into $s-1$ unitary constituents, it follows that the maximum number of types of components of a $(k-1)$-th order interaction in an s^m design equals the number of k-combinations of $s-1$ suffixes when each suffix may be repeated any number of times. This number is known to be (Chrystal, 1931, pp. 10–12)

$$ {}^{s-1}H_k = {}^{s+k-2}C_k = \frac{(s-1)\,s\,(s+1)\ldots(s+k-2)}{\underline{|k}} \tag{11} $$

Consider the general type of unitary component of a $(k-1)$-th order interaction in which the first suffix occurs l_1 times, the second suffix l_2 times; in general, the r-th suffix l_r times; and, finally, the $(s-1)$-th suffix l_{s-1} times, where $l_1, l_2, \ldots, l_r, \ldots, l_{s-1}$ can assume any positive integral values subject to the restriction

$$ \sum_{r=1}^{s-1} l_r = k \tag{12} $$

Then, from what has been stated above, it follows that the number of unitary components of the $(k-1)$-th order interaction belonging to this type is

$$ \frac{\underline{|k}}{\underline{|l_1}\;\underline{|l_2}\cdots\underline{|l_{s-1}}} \tag{13} $$

Then the total number of unitary components of the $(k-1)$-th order interaction is evidently

$$ \Sigma\,\frac{\underline{|k}}{\underline{|l_1}\;\underline{l_2}\cdots\underline{|l_{s-1}}} \tag{14} $$

where the summation is taken over all positive integral values of $l_1, l_2, \ldots, l_{s-1}$ subject to the condition (12).

Now, by the multinomial theorem for a positive integral index, we have

$$(c_1 + c_2 + \ldots + c_{s-1})^k = \Sigma \frac{k\,!}{l_1!\; l_2! \ldots l_{s-1}!} c_1^{\,l_1} c_2^{\,l_2} \ldots c_{s-1}^{\,l_{s-1}} \qquad (15)$$

where $l_1, l_2, \ldots, l_{s-1}$ can take any positive integral values subject to the restriction (12).

Setting $c_1 = c_2 = \ldots = c_{s-1} = 1$ in (15), we obtain

$$(s-1)^k = \Sigma \frac{k!}{l_1!\, l_2! \ldots l_{s-1}!} \qquad (16)$$

It follows from (13) and (16) that all the $(s-1)^k$ degrees of freedom belonging to the $(k-1)$-th order interaction in an s^m design are thus accounted for.

4. Expressions for Unitary Components of the Highest Order Interactions in 3^5, 3^6, 4^4 and 5^3 Designs

We shall now apply the procedure adumbrated in the foregoing Section to obtain expressions for all the possible types of components belonging to the highest order interactions in the case of the 3^5, 3^6, 4^4 and 5^3 designs in terms of sets for these interactions.

(4.1) *Expressions for components of the highest order interaction in 3^5 design.*—The 16 sets, each of 2 degrees of freedom, of the fourth order interaction in a 3^5 design, *viz.*, $(a_0, a_1, a_2) \times (b_0, b_1, b_2) \times \ldots \times (e_0, e_1, e_2)$, denoted by $N_1, N_2, \ldots, N_{16}$ correspond respectively to the 16 parallel pencils of 4-flats in *PG* (5, 3) represented by the equations

$$x_1 + a_{i_2} x_2 + a_{i_3} x_3 + a_{i_4} x_4 + a_{i_5} x_5 = a_t$$
$$(i_2, i_3, i_4, i_5 = 1, 2;\; t = 0, 1, 2) \qquad (17)$$

written in order in the usual systematic manner, $a_0 = 0$, $a_1 = 1$, $a_2 = 2$ being the three elements of *GF* (3). The table showing these sets has, however, been omitted for want of space.

The maximum number of types of unitary components in this case is ${}^2H_5$ or 6. The expressions for these 6 types of components of the fourth order interaction have been worked out and are presented in Table IV.

(4.2) *Expressions for components of the fifth order interaction in 3^6 design.*—The 32 sets, each having two degrees of freedom, of the fifth order interaction in a 3^6 design, *viz.*, $(a_0, a_1, a_2)\times(b_0, b_1, b_2)\times \ldots \times(f_0, f_1, f_2)$, symbolized by $P_1, P_2, \ldots, P_{32}$, correspond respectively to the 32 parallel pencils of 5-flats in $PG(6, 3)$ represented by the equations

$$x_1 + \alpha_{i_2}x_2 + \alpha_{i_3}x_3 + \alpha_{i_4}x_4 + \alpha_{i_5}x_5 + \alpha_{i_6}x_6 = \alpha_t$$
$$(i_2, i_3, \ldots, i_6 = 1, 2;\ t = 0, 1, 2) \tag{18}$$

written in order systematically. As before, the table showing these sets is omitted for lack of space.

The maximum number of types of components in this case is ${}^2H_6$ or 7. The expressions for these 7 types of components of the fifth order interaction have been found and are presented in Table V.

(4.3) *Expressions for components of third order interaction in 4^4 design.*—The 27 sets, each carrying three degrees of freedom, of the third order interaction in a 4^4 design, *viz.*, $(a_0, a_1, a_2, a_3)\times(b_0, b_1, b_2, b_3)\times(c_0, c_1, c_2, c_3)\times(d_0, d_1, d_2, d_3)$, are symbolized by $R_1, R_2, \ldots, R_{27}$, as already described in Section (2.1). These are not given for lack of space.

The maximum number of types of unitary components of the third order interaction is ${}^3H_4$ or 15. The expressions for these 15 types have been derived when the partitioning of the degrees of freedom for a main effect is in accordance with method (ii) of the previous paper, and are given in Table VI.

(4.4) *Expressions for components of second order interaction in 5^3 design.*—The 16 sets, each having four degrees of freedom, of the second order interaction in a 5^3 design, *viz.*, $(a_0, a_1, a_2, a_3, a_4)\times(b_0, b_1, b_2, b_3, b_4)\times(c_0, c_1, c_2, c_3, c_4)$ symbolized by $M_1, M_2, \ldots, M_{16}$, correspond respectively to the 16 parallel pencils of planes in $PG(3, 5)$ represented by the equations

$$x_1 + \alpha_{i_2}x_2 + \alpha_{i_3}x_3 = \alpha_t\ (i_2, i_3 = 1, 2, 3, 4;\ t = 0, 1, 2, 3, 4) \tag{19}$$

$\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$ and $\alpha_4 = 4$ being the five elements of $GF(5)$. These sets are, as before, omitted for want of space.

The partitioning of the 4 degrees of freedom for the first main effect A may now be done as under:

$$\left.\begin{aligned} A_1 &= 2a_4 + a_3 - a_1 - 2a_0 \\ A_2 &= 2a_4 - a_3 - 2a_2 - a_1 + 2a_0 \\ A_3 &= a_4 - 2a_3 + 2a_1 - a_0 \\ A_4 &= a_4 - 4a_3 + 6a_2 - 4a_1 + a_0 \end{aligned}\right\} \qquad (20)$$

and similarly for the other two main effects.

Then the maximum number of types of components of the second order interaction is ${}^4H_3$ or 20, for which the expressions have been presented in Table VII.

5. Reduction in the Number of Possible Types of Components of a $(k-1)$-th Order Interaction in an s^m Design

When the $s-1$ unitary constituents of a main effect are all of distinct types and are not similar in groups of types, the number of possible types of components of a $(k-1)$-th order interaction in an s^m design equals the maximum number of these types, *viz.*, ${}^{s-1}H_k$. When, however, the $s-1$ unitary components of a main effect themselves reduce to q types ($q < s-1$), it has been found that ${}^qH_k$ is the lower bound for the number of distinct types of components of the $(k-1)$-th order interaction.

Consider the components of the second order interaction in a 5^3 design discussed above in Section (4·4) and the expressions for the 20 components given in Table VII. It will be seen that the first and third unitary components in (20) belong to the same type. Consequently, the lower bound for the number of distinct types in this case is 10. It would appear from Table VII that the number of distinct types of components is also actually 10, under which the 20 types of components given there can be classified. The distinct types and the components belonging to each are shown in Table VIII.

Let us now take a 4^3 design and consider first the splitting up of a main effect according to method (ii) of the previous paper. In this case, the first and third unitary components are of the same type. From the expressions for the components of the second order interaction given in Table 6 of the previous paper, it would appear that in this case there are five distinct types of components symbolized by 111, 112, 122, 113 and 222, whilst the lower bound of the number of possible types is ${}^2H_3$ or 4. When, however, the splitting up of the 3 degrees of freedom for a main effect is according to method (i) of the previous paper, all the three unitary constituents are of one and the same type. It would appear from Table V of the previous paper

TABLE VIII

Distinct Types of Components of Second Order Interaction in 5^3 Design

Sl. No.	Distinct types			Components belonging to the same distinct types
1	1	1	1	$A_1\ B_1\ C_1$, $A_3\ B_1\ C_1$, $A_3\ B_3\ C_1$, $A_3\ B_3\ C_3$
2	1	1	2	$A_2\ B_1\ C_1$, $A_3\ B_2\ C_1$, $A_3\ B_3\ C_2$
3	1	2	2	$A_2\ B_2\ C_1$, $A_3\ B_2\ C_2$
4	2	2	2	$A_2\ B_2\ C_2$
5	1	1	4	$A_4\ B_1\ C_1$, $A_4\ B_3\ C_1$, $A_4\ B_3\ C_3$
6	1	4	4	$A_4\ B_4\ C_1$, $A_4\ B_4\ C_3$
7	4	4	4	$A_4\ B_4\ C_4$
8	2	2	4	$A_2\ B_4\ C_2$
9	2	4	4	$A_2\ B_4\ C_4$
10	1	2	4	$A_4\ B_2\ C_1$, $A_4\ B_3\ C_2$

that all the unitary components of the second order interaction belong to one type, the lower bound of the number of possible types of components in this case being also 1. This result is also true for the components of a $(k-1)$-th order interaction in a 4^m design when the splitting up is by method (i). In the case of a 4^4 design, the expression for the component $A'B'C'D'$ of this type has already been worked out above in (7).

Let us finally consider a 4^4 design, expressions for the components of the third order interaction for which are given in Table VI. The lower bound for the number of distinct types of components in this case is 5, whilst the actual number of distinct types is found to be 7 under which all the 15 types of components shown in Table VI can be categorized. The distinct types and the components belonging to each are shown in Table IX.

From the above examples, it is surmised that when, in an s^m design, the sub-division of the $s-1$ degrees of freedom for a main effect results in only q distinct types of unitary constituents, where $q < s-1$, the number of distinct types of components of a $(k-1)$-th order interaction is ${}^qH_k$ when s is a prime number. When, however, s is a power of a prime, ${}^qH_k$ is the lower bound for the number of these

TABLE IX

Distinct Types of Components of Third Order Interaction in 4^4 Design

Sl. No.	Types	Components belonging to the same distinct types
1	1 1 1 1	$A_1 B_1 C_1 D_1$, $A_1 B_1 C_1 D_3$, $A_1 B_3 C_3 D_3$
2	1 1 1 2	$A_1 B_1 C_1 D_2$, $A_1 B_1 C_3 D_2$, $A_1 B_3 C_3 D_3$, $A_3 B_3 C_3 D_3$
3	1 1 2 2	$A_1 B_1 C_2 D_2$, $A_1 B_3 C_2 D_2$, $A_3 B_3 C_2 D_2$
4	1 2 2 2	$A_1 B_2 C_2 D_2$, $A_3 B_2 C_2 D_2$
5	2 2 2 2	$A_2 B_2 C_2 D_2$
6	1 1 3 3	$A_1 B_1 C_3 D_3$
7	3 3 3 3	$A_3 B_3 C_3 D_3$

distinct types, this number being actually equalled in some cases when s is a power of 2.

6. CONCLUDING REMARKS

Given the expression for a type of unitary component of a $(k - 1)$-th order interaction in an s^m design, it has not been possible to evolve a method, if any, by which the expressions for the other components belonging to that type can be derived therefrom. If, therefore, expressions for any components of the highest order interactions in 3^5, 3^6, 4^4 and 5^3 designs other than those given in Tables IV, V, VI and VII are required, these will have to be obtained by the procedure described in Section 2.

7. SUMMARY

The general method for expressing any single degree of freedom for treatments in the general symmetrical factorial design s^m, where s is a prime positive integer or a power of a prime and m any positive integer, in terms of its sets for main effects and interactions developed in the previous paper has been considerably simplified here when the single degree of freedom belongs to a $(k - 1)$-th order interaction (k varying from 1 to m). This simplified method has been utilized for obtaining expressions for single degrees of freedom belonging to the highest order interactions in 3^5, 3^6, 4^4 and 5^3 designs.

Types of components of a $(k - 1)$-th order interaction in an s^m design (k varying from 1 to m) have been defined and it has been

shown that the maximum number of types of these components is ${}^{s-1}H_k$, this upper bound being attained when the $s - 1$ unitary components of each main effect are all of different types. When, however, these reduce to only q distinct types, the lower bound for the number of distinct types of components is ${}^{q}H_k$. It is surmised that this lower bound is always equalled when s is a prime number and also sometimes when s is a power of 2.

It is not known whether a method exists by which expressions for the components of a $(k - 1)$-th order interaction belonging to one type can be derived from the given expression for one such component.

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